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# Control of Flexible Aircraft Executing Time-Dependent Maneuvers

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**A new generation of aircraft, namely, unmanned aerial vehicles (UAVs), and, in particular, autonomous UAVs, are expected to carry out very critical maneuvers, well in excess of what pilots are able to tolerate. A newly developed theory for the dynamics and control of maneuvering flexible aircraft is ideally suited for the analysis and design of such aircraft. The state equations for maneuvering flexible aircraft are nonlinear and of high dimension. With due consideration to the way aircraft are flown, the problem can be separated into one for the generally large aircraft motions on a given flight path and another one for small perturbations from the flight path and elastic vibration, where the second problem receives inputs from the first. The case in which the flight path represents a time-dependent maneuver, so that the system of perturbation equations is time varying is studied. Several developments designed to facilitate the treatment of time-varying systems are provided. In particular, included are 1) an explicit derivation of the matrices defining the state equations, 2) a new approach to the control of the perturbations from the flight path and the elastic vibration, and 3) expressions for the implementation of the controls in discrete time. Computer solutions for the system response can be carried out conveniently by the use of MATLAB® and MATHEMATICA. A numerical example illustrates the approach for the case of a flexible aircraft executing a pitch maneuver.**

## Nomenclature

| | | |
|---|---|---|
| $A^*$, $B^*$ | = | coefficient matrices in first-order state equations |
| $A(t)$ | = | closed-loop system matrix |
| $C_e$ | = | matrix of direction cosines between $x_e y_e z_e$ and $x_f y_f z_f$ |
| $C_f$ | = | matrix of direction cosines between $x_f y_f z_f$ and $XYZ$ |
| $C_w$ | = | matrix of direction cosines between $x_w y_w z_w$ and $x_f y_f z_f$ |
| $C_\eta$ | = | damping matrix |
| $D_m$ | = | matrix relating component generalized coordinate vector $\boldsymbol{q}$ to aircraft generalized coordinate vector $\boldsymbol{\xi}$ |
| $d_i$ | = | drag force per unit span of component $i$ |
| $E_f$ | = | matrix relating Eulerian velocities to angular quasi-velocities |
| $\boldsymbol{F}, \boldsymbol{M}$ | = | resultant of gravity, aerodynamic, propulsion, and control force and moment vectors |
| $\boldsymbol{F}_E$ | = | engine thrust vector |
| $\boldsymbol{F}_{\text{ext}}$ | = | external disturbing force vector |
| $\mathcal{F}^{(1)}$ | = | first-order generalized force vector |
| $\boldsymbol{f}_{ai}, \boldsymbol{f}_{si}$ | = | distributed aerodynamic force vectors for component $i$ |
| $\boldsymbol{f}_{gi}$ | = | distributed gravity force vector for component $i$ |
| $\boldsymbol{f}_i$ | = | distributed force vector for component $i$ due to gravity, aerodynamics, and controls |
| $\boldsymbol{f}_1, \boldsymbol{f}_2$ | = | nonlinear functions of the state vector |
| $G$ | = | control gain matrix for linear quadratic regulator |
| $g$ | = | gravitational constant |
| $H_c$ | = | gain matrix for direct feedback control |
| $H(t)$ | = | matrix of coefficients for aerodynamic, distributed controls, and gravity forces |
| $I$ | = | identity matrix |
| $J$ | = | inertia matrix for the deformed aircraft |
| $K$ | = | stiffness matrix |
| $k_j$, $c_j$ | = | gain coefficients for direct feedback controls |
| $\ell_i$ | = | lift force per unit span of component $i$ |
| $M$ | = | system mass matrix |
| $M_{rr}^{(0)}$ | = | inertia matrix of undeformed aircraft |
| $M_\xi$ | = | reduced mass matrix |
| $m$ | = | total aircraft mass |
| $m_i$ | = | mass of component $i$ |
| $O_i$ | = | origin of body axes for component $i$ |
| $\boldsymbol{p}$ | = | momentum vector for whole aircraft |
| $\boldsymbol{p}_{Vf}, \boldsymbol{p}_{\omega f}$ | = | momentum vectors for aircraft rigid-body translation and rotation |
| $\tilde{p}_{Vf}$, $\tilde{p}_{\omega f}$ | = | skew symmetric matrices derived from $\boldsymbol{p}_{Vf}, \boldsymbol{p}_{\omega f}$ |
| $\boldsymbol{p}_\eta$ | = | generalized momentum vector |
| $Q$, $R$ | = | weighting matrices in performance measure |
| $\boldsymbol{Q}_\xi$ | = | generalized force vector |
| $\boldsymbol{q}_{ui}, \boldsymbol{q}_{\psi i}$ | = | vectors of generalized coordinates for bending and torsion of component $i$ |
| $\boldsymbol{R}_f$ | = | position vector of origin $O_f$ of $x_f y_f z_f$ relative to $XYZ$, $\boldsymbol{R}_f(X_f, Y_f, Z_f)$ |
| $\boldsymbol{r}_{fw}$ | = | radius vector from $O_f$ to $O_w$ |
| $\boldsymbol{r}_i$ | = | nominal position vector of point on component $i$ |
| $\tilde{S}$ | = | matrix of first moments of inertia of deformed aircraft |
| $s_i$ | = | side force per unit span of component $i$ |
| $T$ | = | kinetic energy; sampling period |
| $t_k$ | = | sampling time |
| $\boldsymbol{u}$ | = | control vector |
| $\boldsymbol{V}_f, \boldsymbol{\omega}_f$ | = | translational and angular quasi-velocity vectors of $x_f y_f z_f$ |
| $\tilde{V}_f$, $\tilde{\omega}_f$ | = | skew symmetric matrices derived from $\boldsymbol{V}_f, \boldsymbol{\omega}_f$ |

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|---|---|---|
| $\bar{V}_i(\boldsymbol{r}_i, t)$ | = | velocity vector of point on component $i$ |
| $XYZ$ | = | inertial axes |
| $x_j y_i z_i$ | = | body axes of component $i$ |
| $\boldsymbol{x}(k)$ | = | state vector at $t_k = kT$ |
| $\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}$ | = | zero- and first-order state vectors |
| $\alpha_i$ | = | angle of attack of lift force for component $i$ |
| $\beta_i$ | = | angle of attack of side force for component $i$ |
| $\delta_a^{(0)}, \delta_a^{(1)}$ | = | zero- and first-order aileron angles |
| $\delta_e^{(0)}, \delta_e^{(1)}$ | = | zero- and first-order elevator angles |
| $\delta_r^{(0)}, \delta_r^{(1)}$ | = | zero- and first-order rudder angles |
| $\delta(\boldsymbol{r} - \boldsymbol{r}_E)$ | = | spatial Dirac delta function at $\boldsymbol{r} = \boldsymbol{r}_E$ |
| $\boldsymbol{\eta}$ | = | generalized velocity vector |
| $\boldsymbol{\theta}_f$ | = | symbolic vector of Eulerian angles between $x_f y_f z_f$ and $XYZ$ |
| $\boldsymbol{\xi}$ | = | generalized displacement vector |
| $\rho_i$ | = | mass density of component $i$ |
| $\Phi_{ui}, \Phi_{\psi i}$ | = | matrices of shape functions for bending and torsion of component $i$ |
| $\psi, \theta, \phi$ | = | Eulerian angles from inertial axes $XYZ$ to body axes $x_f y_f z_f$ |

## I. Introduction

THE subject of dynamics and control of flexible aircraft has received scant attention over the years, which can be attributed to the complexity of the problem. The motion of flexible aircraft can be described by rigid-body translations and rotations of a given reference frame and elastic deformations of the flexible aircraft components relative to the reference frame. The rigid-body motions tend to be large and are described by nonlinear ordinary differential equations. On the other hand, the elastic deformations tend to be small and are described by boundary-value problems; they involve linear partial differential equations. For practical reasons, the boundary-value problems must be approximated by generally large sets of ordinary differential equations through spatial discretization. The solution of large sets of nonlinear equations has tended to discourage even the most ambitious investigators, particularly in the early days in which powerful computers were not readily available. As a result, studies have tended to focus on certain aspects of the problem under limiting assumptions. At one end of the spectrum, we find flight dynamics,[1] which is concerned for the most part with large motions of rigid aircraft. At the other end, we find aeroelasticity,[2] which is concerned primarily with elastic cantilevered wings.

Although as long ago as six decades there have been some calls for simultaneous consideration of both rigid-body motions and elastic deformations, very few investigations have actually heeded the calls. Moreover, most investigations that have heeded the calls represent technical reports and conference presentations, rather than refereed archive publications. Among these, we single out a comprehensive report by Dusto et al.,[3] representing one of the first serious efforts to include structural and aerodynamic effects in a single formulation. However, the use of two unrelated reference frames, one for the structure and the other for the aerodynamics, raises questions about the validity of the results. Moreover, the investigation is limited to stability statements. In a quite different approach, Fornasier et al.[4] studied fluid–structure interactions by developing a computer program making two independently developed computer codes, one for aerodynamics and the other for structural mechanics, work together. Worthy of notice are several time simulations of the response. Much more satisfactory was the approach used by Waszak and Schmidt[5] to derive Lagrange's equations of motion for flexible aircraft. The model was used for a parametric study of the flexibility effects.

A new generation of aircraft, namely, unmanned aerial vehicles (UAVs), and, in particular, autonomous UAVs, are designed to carry out very critical maneuvers, well in excess of what pilots are able to tolerate. To accommodate such aircraft, a new theory for the dynamics and control of maneuvering flexible aircraft is necessary. Quite recently, Meirovitch and Tuzcu[6,7] have developed just such a theory. Based on fundamental principles, the theory integrates seamlessly pertinent material from analytical dynamics, structural dynamics, aerodynamics, and controls into a single formulation. The unified formulation includes automatically both aircraft rigid-body motions and elastic deformations, as well as aerodynamic, gravity, propulsion, and control forces, in addition to forces of an external nature, such as gusts. With due consideration to the way in which aircraft are flown, a perturbation approach is used to separate the problem into two parts, one for the translations and rotations of an aircraft reference frame, referred to as quasi-rigid flight dynamics, and one for small perturbations in the rigid-body variables and elastic vibration, referred to as an extended perturbation problem. According to the perturbation theory, the second problem receives inputs from the first, but the solution of the second problem does not enter into the solution of the first. This is not to be interpreted as implying that the elastic deformations do not affect the aircraft rigid-body motions. Indeed, the extended perturbation problem contains not only the elastic variables but also perturbations in the rigid-body variables. The total rigid-body motions include contributions from both the flight dynamics problem and the extended perturbation problem, so that the elastic deformations do affect the aircraft rigid-body motions, albeit in a small way only. It is the task of the feedback controls to ensure that both the elatic displacements and the perturbations in the ridig-body variables remain small, and go to zero with time. A numerical example illustrates the theory for two flight cases, steady level cruise and a level steady turn maneuver.

The extended perturbation problem is linear, albeit of high dimension. When the input from the flight dynamics is constant, the system is time invariant. This is the case studied in Refs. 6 and 7. When the aircraft executes a time-dependent maneuver, the inputs from the flight dynamics depend on time, so that the system is time varying. This paper is concerned with just such time-varying systems. It contains several new developments, all designed to better handle time-varying problems. In particular, it contains an explicit derivation of the matrices defining the state equations, showing clearly the inputs from the time-dependent maneuver. Moreover, it includes a new control design consisting of a combination of the linear quadratic regulator (LQR) and the direct feedback control methods.[8] To generate computer solutions for time-varying systems in discrete time, corresponding recursive equations are derived. Computer solutions of the closed-loop state equations for the system response can be carried out conveniently by the use of MATLAB® and MATHEMATICA. A numerical example illustrates the approach for a time-dependent maneuver representing the transition from a rectilinear flight to a pitch maneuver and back to rectilinear flight. A number of simulations of flexible aircraft response in discrete time are included. The computer code uses MATHEMATICA.

## II. Quasi-Rigid Aircraft Maneuvers

The state equations describing the dynamics and control of maneuvering flexible aircraft are nonlinear and of high dimension, in which the nonlinearity is due to the rigid-body variables and the high dimensionality is due to the elastic deformations. Ideally, the aircraft flies along a given path as if it were rigid. Because of various factors, however, the aircraft undergoes small elastic vibration and perturbations in the rigid-body motions. Under the assumption that the quantities defining the flight path are much larger than those defining the elastic variables and the perturbations in the rigid-body variables, a perturbation approach permits separation of the problem into one for the large rigid-body variables and another one for the small perturbations in the rigid-body variables and the elastic variables. As indicated in the Introduction, the first is referred to as a quasi-rigid flight dynamics problem and the second as an extended perturbation problem.

From Ref. 6, the quasi-rigid flight dynamics equations, or the zero-order state equations, are

$$\dot{\boldsymbol{R}}_f^{(0)} = C_f^{(0)T} \boldsymbol{V}_f^{(0)}, \qquad \dot{\boldsymbol{\theta}}_f^{(0)} = \left(E_f^{(0)}\right)^{-1} \boldsymbol{\omega}_f^{(0)}$$

$$\dot{\boldsymbol{p}}_{Vf}^{(0)} = -\tilde{\omega}_f^{(0)} \boldsymbol{p}_{Vf}^{(0)} + \boldsymbol{F}^{(0)}, \qquad \dot{\boldsymbol{p}}_{\omega f}^{(0)} = -\tilde{V}_f^{(0)} \boldsymbol{p}_{Vf}^{(0)} - \tilde{\omega}_f^{(0)} \boldsymbol{p}_{\omega f}^{(0)} + \boldsymbol{M}^{(0)} \tag{1}$$

where the superscript (0) denotes zero-order quantities, obtained by regarding the aircraft as rigid. In particular, $\boldsymbol{R}_f^{(0)}$ is the position of the origin $O_f$ of the body axes $x_f y_f z_f$ (Fig. 1 of Ref. 6); $\boldsymbol{\theta}_f^{(0)}$ is a symbolic vector of Eulerian angles between $x_f y_f z_f$ and the inertial axes $XYZ$, both $\boldsymbol{R}_f^{(0)}$ and $\boldsymbol{\theta}_f^{(0)}$ being in terms of inertial axes components; $\boldsymbol{V}_f^{(0)}$ and $\boldsymbol{\omega}_f^{(0)}$ are aircraft quasi-velocity vectors in translation and rotation, that is, velocity vectors in terms of body axes components; $C_f^{(0)}$ is a matrix of direction cosines between $x_f y_f z_f$ and $XYZ$ and $E_f^{(0)}$ is a matrix relating $\boldsymbol{\omega}_f^{(0)}$ to $\boldsymbol{\theta}_f^{(0)}$, both matrices being given in Ref. 6; $\boldsymbol{p}_{Vf}^{(0)}$ and $\boldsymbol{p}_{\omega f}^{(0)}$ are momentum vectors in translation and rotation in terms of body axes components; $\tilde{V}_f^{(0)}$ and $\tilde{\omega}_f^{(0)}$ are skew symmetric matrices derived from $\boldsymbol{V}_f^{(0)}$ and $\boldsymbol{\omega}_f^{(0)}$ (Ref. 6); and $\boldsymbol{F}^{(0)}$ and $\boldsymbol{M}^{(0)}$ are resultants of zero-order gravity, aerodynamic, propulsion, and control generalized force and moment vectors, respectively. The latter have the expressions

$$\boldsymbol{F}^{(0)} = \int \left[\boldsymbol{f}_f^{(0)} + \boldsymbol{F}_E^{(0)}\delta(\boldsymbol{r}_f - \boldsymbol{r}_E)\right] \mathrm{d}D_f + \sum_{i=w,e} C_i^T \int \boldsymbol{f}_i^{(0)}\,\mathrm{d}D_i$$

$$\boldsymbol{M}^{(0)} = \int \tilde{r}_f\left[\boldsymbol{f}_f^{(0)} + \boldsymbol{F}_E^{(0)}\delta(\boldsymbol{r}_f - \boldsymbol{r}_E)\right] \mathrm{d}D_f + \sum_{i=w,e} \int \left(\tilde{r}_{fi} C_i^T + C_i^T \tilde{r}_i\right)\boldsymbol{f}_i^{(0)}\,\mathrm{d}D_i \tag{2}$$

in which $\boldsymbol{f}_f^{(0)}$, $\boldsymbol{f}_w^{(0)}$, and $\boldsymbol{f}_e^{(0)}$ are actual distributed zero-order force vectors acting on the individual components; their expressions are also given in Ref. 6. Moreover, $\boldsymbol{F}_E^{(0)}$ is the engine thrust vector. Equations (1) contain both momenta and velocities, so that they must be solved in conjunction with the zero-order momenta–velocities relation. From Ref. 6, in the absence of elastic deformations, this relation can be written as

$$\boldsymbol{p}_{Vf}^{(0)} = m\boldsymbol{V}_f^{(0)} + \tilde{S}^{(0)T}\boldsymbol{\omega}_f^{(0)}, \qquad \boldsymbol{p}_{\omega f}^{(0)} = \tilde{S}^{(0)}\boldsymbol{V}_f^{(0)} + J^{(0)}\boldsymbol{\omega}_f^{(0)} \tag{3}$$

where $m$ is the total mass, $\tilde{S}^{(0)}$ the skew symmetric matrix of first moments of inertia, and $J^{(0)}$ the inertia matrix, all for the undeformed aircraft.

The state equations can be expressed in a more compact form. To this end, we rewrite Eqs. (1) and (3) as follows:

$$\begin{bmatrix} \dot{\boldsymbol{R}}_f^{(0)} \\ \dot{\boldsymbol{\theta}}_f^{(0)} \end{bmatrix} = \begin{bmatrix} C_f^{(0)T} & 0 \\ 0 & \left(E_f^{(0)}\right)^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_f^{(0)} \\ \boldsymbol{\omega}_f^{(0)} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\boldsymbol{p}}_{Vf}^{(0)} \\ \dot{\boldsymbol{p}}_{\omega f}^{(0)} \end{bmatrix} = \begin{bmatrix} 0 & \tilde{p}_{Vf}^{(0)} \\ \tilde{p}_{Vf}^{(0)} & \tilde{p}_{\omega f}^{(0)} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_f^{(0)} \\ \boldsymbol{\omega}_f^{(0)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{F}^{(0)} \\ \boldsymbol{M}^{(0)} \end{bmatrix} \tag{4}$$

and

$$\begin{bmatrix} \boldsymbol{p}_{Vf}^{(0)} \\ \boldsymbol{p}_{\omega f}^{(0)} \end{bmatrix} = \begin{bmatrix} mI & \tilde{S}^{(0)T} \\ \tilde{S}^{(0)} & J^{(0)} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_f^{(0)} \\ \boldsymbol{\omega}_f^{(0)} \end{bmatrix} \tag{5}$$

Then, introducing the zero-order state vector $\boldsymbol{x}^{(0)} = [\boldsymbol{R}_f^{(0)T} \;\; \boldsymbol{\theta}_f^{(0)T} \;\; \boldsymbol{p}_{Vf}^{(0)T} \;\; \boldsymbol{p}_{\omega f}^{(0)T}]^T$, Eqs. (4) and (5) can be combined into

$$\dot{\boldsymbol{x}}^{(0)}(t) = \boldsymbol{f}_1\left(\boldsymbol{x}^{(0)}(t)\right) + \boldsymbol{f}_2\left(\boldsymbol{x}^{(0)}(t)\right) + B^{(0)}\left(\boldsymbol{x}^{(0)}(t)\right)\boldsymbol{u}^{(0)}(t) \tag{6}$$

where

$$\boldsymbol{f}_1\left(\boldsymbol{x}^{(0)}(t)\right) = \left[\begin{array}{c|c} 0 & \begin{bmatrix} C_f^{(0)T} & 0 \\ 0 & \left(E_f^{(0)}\right)^{-1} \end{bmatrix} \left(M_{rr}^{(0)}\right)^{-1} \\ \hline 0 & \begin{bmatrix} 0 & \tilde{p}_{Vf}^{(0)} \\ \tilde{p}_{Vf}^{(0)} & \tilde{p}_{\omega f}^{(0)} \end{bmatrix} \left(M_{rr}^{(0)}\right)^{-1} \end{array}\right] \boldsymbol{x}^{(0)} \tag{7}$$

is a nonlinear vector function of the state, in which

$$M_{rr}^{(0)} = \begin{bmatrix} mI & \tilde{S}^{(0)T} \\ \tilde{S}^{(0)} & J^{(0)} \end{bmatrix} \tag{8}$$

is the inertia matrix of the undeformed aircraft; $\boldsymbol{f}_2$ is a force vector due to aerodynamics and gravity; $B^{(0)}$ is a coefficient matrix with the top one-half equal to the null matrix; and $\boldsymbol{u}^{(0)} = [F_E^{(0)} \;\; \delta_a^{(0)} \;\; \delta_e^{(0)} \;\; \delta_r^{(0)}]^T$ is the zero-order control vector, in which $F_E^{(0)}$ is the engine thrust and $\delta_a^{(0)}$, $\delta_e^{(0)}$, and $\delta_r^{(0)}$ are the aileron, elevator, and rudder angles, respectively.

## III. Perturbation State Equations

The perturbation state equations are linear, but they contain inputs from the quasi-rigid flight dynamics problem. For control design and computer simulation purposes, it is convenient to cast the equations in vector-matrix form. Although this form was given in Ref. 7, the explicit expression for the system matrix $A(t)$ was not produced there. Yet, for efficient computer coding, such as in MATLAB, it is necessary to write the state equations in standard form,[8] which involves the matrix $A(t)$ explicitly. In this section, we derive the state equations in standard form.

From Ref. 7, the deviations from the aircraft flight path are described by the perturbation state equations, or the first-order state equations,

$$\dot{\boldsymbol{R}}_f^{(1)} = C_f^{(0)T}\boldsymbol{V}_f^{(1)} + C_f^{(1)T}\boldsymbol{V}_f^{(0)}$$

$$\dot{\boldsymbol{\theta}}_f^{(1)} = \left(E_f^{(0)}\right)^{-1}\boldsymbol{\omega}_f^{(1)} - \left(E_f^{(0)}\right)^{-1} E_f^{(1)} \left(E_f^{(0)}\right)^{-1}\boldsymbol{\omega}_f^{(0)}, \qquad \dot{\boldsymbol{\xi}} = \boldsymbol{\eta}$$

$$\dot{\boldsymbol{p}}_{Vf}^{(1)} = -\tilde{\omega}_f^{(1)}\boldsymbol{p}_{Vf}^{(0)} - \tilde{\omega}_f^{(0)}\boldsymbol{p}_{Vf}^{(1)} + \boldsymbol{F}^{(1)}$$

$$\dot{\boldsymbol{p}}_{\omega f}^{(1)} = -\tilde{V}_f^{(1)}\boldsymbol{p}_{Vf}^{(0)} - \tilde{V}_f^{(0)}\boldsymbol{p}_{Vf}^{(1)} - \tilde{\omega}_f^{(1)}\boldsymbol{p}_{\omega f}^{(0)} - \tilde{\omega}_f^{(0)}\boldsymbol{p}_{\omega f}^{(1)} + \boldsymbol{M}^{(1)}$$

$$\dot{\boldsymbol{p}}_\eta = \left(\frac{\partial T}{\partial \boldsymbol{\xi}}\right)^{(1)} - K_\xi\boldsymbol{\xi} - C_\eta\boldsymbol{\eta} + \left(\frac{\partial T}{\partial \boldsymbol{\xi}}\right)^{(0)} + \boldsymbol{Q}_\xi \tag{9}$$

in which

$$C_f^{(1)} = \left.\frac{\partial C_f}{\partial \psi}\right|_{\boldsymbol{\theta}_f^{(0)}}\psi^{(1)} + \left.\frac{\partial C_f}{\partial \theta}\right|_{\boldsymbol{\theta}_f^{(0)}}\theta^{(1)} + \left.\frac{\partial C_f}{\partial \phi}\right|_{\boldsymbol{\theta}_f^{(0)}}\phi^{(1)} = C_{f\psi}^{(0)}\psi^{(1)} + C_{f\theta}^{(0)}\theta^{(1)} + C_{f\phi}^{(0)}\phi^{(1)}$$

$$E_f^{(1)} = \left.\frac{\partial E_f}{\partial \theta}\right|_{\boldsymbol{\theta}_f^{(0)}}\theta^{(1)} + \left.\frac{\partial E_f}{\partial \phi}\right|_{\boldsymbol{\theta}_f^{(0)}}\phi^{(1)} = E_{f\theta}^{(0)}\theta^{(1)} + E_{f\phi}^{(0)}\phi^{(1)} \tag{10}$$

where

$$C_{f\psi}^{(0)} = \left.\frac{\partial C_f}{\partial \psi}\right|_{\boldsymbol{\theta}_f^{(0)}}, \qquad C_{f\theta}^{(0)} = \left.\frac{\partial C_f}{\partial \theta}\right|_{\boldsymbol{\theta}_f^{(0)}}, \qquad C_{f\phi}^{(0)} = \left.\frac{\partial C_f}{\partial \phi}\right|_{\boldsymbol{\theta}_f^{(0)}}$$

$$E_{f\theta}^{(0)} = \left.\frac{\partial E_f}{\partial \theta}\right|_{\boldsymbol{\theta}_f^{(0)}}, \qquad E_{f\phi}^{(0)} = \left.\frac{\partial E_f}{\partial \phi}\right|_{\boldsymbol{\theta}_f^{(0)}} \tag{11}$$

in which $\boldsymbol{\theta}_f^{(1)} = [\psi^{(1)} \;\; \theta^{(1)} \;\; \phi^{(1)}]^T$ is a vector of perturbations in the Eulerian angles $\psi, \theta$, and $\phi$ between axes $x_f y_f z_f$ and $XYZ$. Introducing the notation

$$C_{fV}^T = \left[C_{f\psi}^{(0)T}\boldsymbol{V}_f^{(0)} \quad C_{f\theta}^{(0)T}\boldsymbol{V}_f^{(0)} \quad C_{f\phi}^{(0)T}\boldsymbol{V}_f^{(0)}\right]$$

$$E_{f\omega} = \left[0 \quad \left(E_f^{(0)}\right)^{-1} E_{f\theta}^{(0)}\left(E_f^{(0)}\right)^{-1}\boldsymbol{\omega}_f^{(0)} \quad \left(E_f^{(0)}\right)^{-1} E_{f\phi}^{(0)}\left(E_f^{(0)}\right)^{-1}\boldsymbol{\omega}_f^{(0)}\right] \tag{12}$$

we can write

$$C_f^{(1)T}\boldsymbol{V}_f^{(0)} = C_{fV}^T\boldsymbol{\theta}_f^{(1)}, \qquad \left(E_f^{(0)}\right)^{-1} E_f^{(1)} \left(E_f^{(0)}\right)^{-1}\boldsymbol{\omega}_f^{(0)} = E_{f\omega}\boldsymbol{\theta}_f^{(1)} \tag{13}$$

Moreover, the first-order generalized forces have the expressions

$$\boldsymbol{F}^{(1)} = \int \left[\boldsymbol{f}_f^{(1)} + \boldsymbol{F}_E^{(1)}\delta(\boldsymbol{r}_f - \boldsymbol{r}_E)\right] \mathrm{d}D_f + \sum_{i=w,e} C_i^T \int \boldsymbol{f}_i^{(1)}\,\mathrm{d}D_i$$

$$\boldsymbol{M}^{(1)} = \int \left\{\tilde{r}_f\left[\boldsymbol{f}_f^{(1)} + \boldsymbol{F}_E^{(1)}\delta(\boldsymbol{r}_f - \boldsymbol{r}_E)\right] + (\widetilde{\Phi_{uf} D_{uf}\boldsymbol{\xi}})\left[\boldsymbol{f}_f^{(0)} + \boldsymbol{F}_E^{(0)}\delta(\boldsymbol{r}_f - \boldsymbol{r}_E)\right]\right\} \mathrm{d}D_f + \sum_{i=w,e}\int\left\{\left(\tilde{r}_{fi}C_i^T + C_i^T\tilde{r}_i\right)\boldsymbol{f}_i^{(1)} + \left[(\widetilde{\Phi_{ufi}D_{uf}\boldsymbol{\xi}})C_i^T + C_i^T(\widetilde{\Phi_{ui}D_{ui}\boldsymbol{\xi}})\right]\boldsymbol{f}_i^{(0)}\right\}\mathrm{d}D_i$$

$$\boldsymbol{Q}_\xi = \int (\Phi_{uf}D_{uf} + \tilde{r}_f^T\Phi_{\psi f}D_{\psi f})^T\left[\boldsymbol{f}_f^{(0)} + \boldsymbol{f}_f^{(1)} + \left(\boldsymbol{F}_E^{(0)} + \boldsymbol{F}_E^{(1)}\right)\times\delta(\boldsymbol{r}_f - \boldsymbol{r}_E)\right]\mathrm{d}D_f + \sum_{i=w,e}\int\left[\left(\tilde{r}_i^T C_i\Delta\boldsymbol{\Phi}_{ufi} + C_i\Phi_{ufi}\right)D_{uf} + \Phi_{ui}D_{ui} + \left(\tilde{r}_i^T C_i + C_i\tilde{r}_{fi}^T\right)\Phi_{\psi fi}D_{\psi f} + \tilde{r}_i^T\Phi_{\psi i}D_{\psi i}\right]^T\left(\boldsymbol{f}_i^{(0)} + \boldsymbol{f}_i^{(1)}\right)\mathrm{d}D_i \tag{14}$$

where $\boldsymbol{f}_f^{(1)}$, $\boldsymbol{f}_w^{(1)}$, and $\boldsymbol{f}_e^{(1)}$ are component-distributed first-order forces and $D_{uf}, D_{uw}, \ldots, D_{\psi e}$ are submatrices of the matrix $D_m$ (Ref. 7) relating the component generalized coordinate vectors $\boldsymbol{q}_{uf}, \boldsymbol{q}_{uw}, \ldots, \boldsymbol{q}_{\psi e}$ to the $m$-dimensional aircraft generalized coordinate vector $\boldsymbol{\xi}$, as follows:

$$\boldsymbol{q} = \begin{bmatrix}\boldsymbol{q}_{uf}\\ \hline \boldsymbol{q}_{uw}\\ \hline \vdots \\ \hline \boldsymbol{q}_{\psi e}\end{bmatrix} = \begin{bmatrix}D_{uf}\\ \hline D_{uw}\\ \hline \vdots \\ \hline D_{\psi e}\end{bmatrix}\boldsymbol{\xi} = D_m\boldsymbol{\xi} \tag{15}$$

The kinetic energy can be expressed in the form

$$T = \frac{1}{2}\sum_{i=f,w,e}\int\left(\bar{\boldsymbol{V}}_i^{(0)} + \bar{\boldsymbol{V}}_i^{(1)}\right)^T\left(\bar{\boldsymbol{V}}_i^{(0)} + \bar{\boldsymbol{V}}_i^{(1)}\right)\mathrm{d}m_i \tag{16}$$

where

$$\bar{\boldsymbol{V}}_f^{(0)} = \boldsymbol{V}_f^{(0)} + \tilde{r}_f^T\boldsymbol{\omega}_f^{(0)}$$

$$\bar{\boldsymbol{V}}_f^{(1)} = \boldsymbol{V}_f^{(1)} + \tilde{r}_f^T\boldsymbol{\omega}_f^{(1)} + (\widetilde{\Phi_{uf}D_{uf}\boldsymbol{\xi}})^T\boldsymbol{\omega}_f^{(0)} + \left(\Phi_{uf}D_{uf} + \tilde{r}_f^T\Phi_{\psi f}D_{\psi f}\right)\boldsymbol{\eta}$$

$$\bar{\boldsymbol{V}}_i^{(0)} = C_i\boldsymbol{V}_f^{(0)} + \left(C_i\tilde{r}_{fi}^T + \tilde{r}_i^T C_i\right)\boldsymbol{\omega}_f^{(0)}, \qquad i = w, e$$

$$\bar{\boldsymbol{V}}_i^{(1)} = C_i\boldsymbol{V}_f^{(1)} + \left(C_i\tilde{r}_{fi}^T + \tilde{r}_i^T C_i\right)\boldsymbol{\omega}_f^{(1)} + [C_i(\widetilde{\Phi_{ufi}D_{uf}\boldsymbol{\xi}})^T + (\widetilde{\Phi_{ui}D_{ui}\boldsymbol{\xi}})^T C_i]\boldsymbol{\omega}_f^{(0)} + \left[\left(\tilde{r}_i^T C_i\Delta\Phi_{ufi} + C_i\Phi_{ufi}\right)D_{uf} + \Phi_{ui}D_{ui} + \left(\tilde{r}_i^T C_i + C_i\tilde{r}_{fi}^T\right)\Phi_{\psi fi}D_{\psi f} + \tilde{r}_i^T\Phi_{\psi i}D_{\psi i}\right]\boldsymbol{\eta}, \qquad i = w, e \tag{17}$$

in which an overbar indicates the total velocity of a typical point in the individual component. It follows that

$$\left(\frac{\partial T}{\partial\boldsymbol{\xi}}\right)^{(0)} = D_{uf}^T\left\{\int\Phi_{uf}^T\tilde{\omega}_f^{(0)T}\bar{\boldsymbol{V}}_f^{(0)}\,\mathrm{d}m_f + \sum_{i=w,e}\Phi_{ufi}\tilde{\omega}_f^{(0)T}C_i^T\int\bar{\boldsymbol{V}}_i^{(0)}\,\mathrm{d}m_i\right\} + \sum_{i=w,e}D_{ui}^T\int\Phi_{ui}^T\left(\widetilde{C_i\boldsymbol{\omega}_f^{(0)}}\right)^T\bar{\boldsymbol{V}}_i^{(0)}\,\mathrm{d}m_i$$

$$\left(\frac{\partial T}{\partial\boldsymbol{\xi}}\right)^{(1)} = D_{uf}^T\left\{\int\Phi_{uf}^T\left(\tilde{\omega}_f^{(0)T}\bar{\boldsymbol{V}}_f^{(1)} + \tilde{\omega}_f^{(1)T}\bar{\boldsymbol{V}}_f^{(0)}\right)\mathrm{d}m_f + \sum_{i=w,e}\Phi_{ufi}^T\left[\tilde{\omega}_f^{(0)T}C_i^T\int\bar{\boldsymbol{V}}_i^{(1)}\,\mathrm{d}m_i + \tilde{\omega}_f^{(1)T}C_i^T\int\bar{\boldsymbol{V}}_i^{(0)}\,\mathrm{d}m_i\right]\right\} + \sum_{i=w,e}D_{ui}^T\int\Phi_{ui}^T\left[\left(\widetilde{C_i\boldsymbol{\omega}_f^{(0)}}\right)^T\bar{\boldsymbol{V}}_i^{(1)} + \left(\widetilde{C_i\boldsymbol{\omega}_f^{(1)}}\right)^T\bar{\boldsymbol{V}}_i^{(0)}\right]\mathrm{d}m_i \tag{18}$$

Equations (9) contain both momenta and velocities. To eliminate the velocities, we must use the first-order momenta–velocities relation

$$\boldsymbol{p}^{(1)} = M_\xi^{(0)}\boldsymbol{V}_\eta^{(1)} + M_\xi^{(1)}\boldsymbol{V}_\eta^{(0)} \tag{19}$$

in which $\boldsymbol{p}^{(1)} = [\boldsymbol{p}_{Vf}^{(1)T}\ \ \boldsymbol{p}_{\omega f}^{(1)T}\ \ \boldsymbol{p}_\eta^T]^T$ is the first-order momentum vector, $\boldsymbol{V}_\eta^{(0)} = [\boldsymbol{V}_f^{(0)T}\ \ \boldsymbol{\omega}_f^{(0)T}\ \ \boldsymbol{0}^T]^T$ the $(6+m)$-dimensional zero-order velocity vector, $\boldsymbol{V}_\eta^{(1)} = [\boldsymbol{V}_f^{(1)T}\ \ \boldsymbol{\omega}_f^{(1)T}\ \ \boldsymbol{\eta}^T]^T$ the first-order velocity vector and

$$M_\xi^{(0)} = \left[\begin{array}{cc|c} mI & \tilde{S}^{(0)T} & \\ & & M_{re}^{(0)}D_m \\ \tilde{S}^{(0)} & J^{(0)} & \\ \hline \multicolumn{2}{c|}{D_m^T M_{re}^{(0)T}} & D_m^T M_{ee}^{(0)}D_m\end{array}\right]$$

$$M_\xi^{(1)} = \left[\begin{array}{cc|c} 0 & \tilde{S}^{(1)T} & \\ & & M_{re}^{(1)}D_m \\ \tilde{S}^{(1)} & J^{(1)} & \\ \hline \multicolumn{2}{c|}{D_m^T M_{re}^{(1)T}} & D_m^T M_{ee}^{(1)}D_m\end{array}\right] \tag{20}$$

are the $(6+m)\times(6+m)$ zero-order and first-order mass matrices.

Next, we cast the state equations, Eqs. (9), in standard form. To this end, we first solve Eq. (19) for $\boldsymbol{V}_\eta^{(1)}$ and write

$$\boldsymbol{V}_\eta^{(1)} = \left(M_\xi^{(0)}\right)^{-1}\left(\boldsymbol{p}^{(1)} - M_\xi^{(1)}\boldsymbol{V}_\eta^{(0)}\right) \tag{21}$$

Then, observing from the second of Eqs. (20) that $M_\xi^{(1)}$ is a linear function of the component generalized coordinates, we have

$$M_\xi^{(1)}\boldsymbol{V}_\eta^{(0)} = \sum_{i=1}^{r}\frac{\partial}{\partial q_{ufi}}M_\xi^{(1)}\boldsymbol{V}_\eta^{(0)}q_{ufi} + \sum_{j=1}^{s}\frac{\partial}{\partial q_{uwj}}M_\xi^{(1)}\boldsymbol{V}_\eta^{(0)}q_{uwj} + \sum_{k=1}^{t}\frac{\partial}{\partial q_{uek}}M_\xi^{(1)}\boldsymbol{V}_\eta^{(0)}q_{uek}$$
$$= \left[M_\xi^{(1)}\big|_{q_u=e_1}\boldsymbol{V}_\eta^{(0)}\quad M_\xi^{(1)}\big|_{q_u=e_2}\boldsymbol{V}_\eta^{(0)}\cdots M_\xi^{(1)}\big|_{q_u=e_{r+s+t}}\boldsymbol{V}_\eta^{(0)}\right]\boldsymbol{q}_u$$
$$= M_{\xi V}\boldsymbol{q}_u = M_{\xi V}U_u\left[\boldsymbol{R}_f^{(1)T}\quad\boldsymbol{\theta}_f^{(1)T}\quad\boldsymbol{\xi}\right]^T \tag{22}$$

where

$$M_{\xi V} = \left[M_\xi^{(1)}\big|_{q_u=e_1}\boldsymbol{V}_\eta^{(0)}\quad M_\xi^{(1)}\big|_{q_u=e_2}\boldsymbol{V}_\eta^{(0)}\cdots M_\xi^{(1)}\big|_{q_u=e_{r+s+t}}\boldsymbol{V}_\eta^{(0)}\right] \tag{23}$$

in which $\boldsymbol{e}_i = [0\ \ 0\ \ \cdots\ \ 1\ \ \cdots\ \ 0]^T$ is a unit vector with 1 in position $i$. Moreover,

$$U_u = [0 \,\vert\, D_m], \qquad D_m = [D_u \,\vert\, D_\psi] \tag{24}$$

in which 0 is a null matrix with six columns and as many rows as the number of bending degrees of freedom of the aircraft.

Next, we insert Eqs. (17) into Eq. (18) and rearrange the result to obtain

$$\left(\frac{\partial T}{\partial\boldsymbol{\xi}}\right)^{(1)} = A_{T\xi}\boldsymbol{\xi} + A_{TV}\boldsymbol{V}_f^{(1)} + A_{T\omega}\boldsymbol{\omega}_f^{(1)} + A_{T\eta}\boldsymbol{\eta} \tag{25}$$

where

$$A_{T\xi} = D_{uf}^T \int \Phi_{uf}^T \tilde{\omega}_f^{(0)T} \tilde{\omega}_f^{(0)} \Phi_{uf} D_{uf} \, \mathrm{d}m_f + \sum_{i=w,e} \int \Big\{ D_{uf}^T \Phi_{ufi}^T \tilde{\omega}_f^{(0)T} \big[ \tilde{\omega}_f^{(0)} \Phi_{ufi} D_{uf} + C_i^T \big(\widetilde{C_i \boldsymbol{\omega}_f^{(0)}}\big) \Phi_{ui} D_{ui} \big] + D_{ui}^T \Phi_{ui}^T \big(\widetilde{C_i \boldsymbol{\omega}_f^{(0)}}\big)^T \big[ C_i \tilde{\omega}_f^{(0)} \Phi_{ufi} D_{uf} + \big(\widetilde{C_i \boldsymbol{\omega}_f^{(0)}}\big) \Phi_{ui} D_{ui} \big] \Big\} \, \mathrm{d}m_i$$

$$A_{TV} = D_{uf}^T \int \Phi_{uf}^T \tilde{\omega}_f^{(0)T} dm_f + \sum_{i=w,e} \int \big[ D_{uf}^T \Phi_{ufi}^T \tilde{\omega}_f^{(0)T} + D_{ui}^T \Phi_{ui}^T \big(\widetilde{C_i \boldsymbol{\omega}_f^{(0)}}\big)^T C_i \big] \, \mathrm{d}m_i$$

$$A_{T\omega} = D_{uf}^T \int \Phi_{uf}^T \big[ \tilde{\omega}_f^{(0)T} \tilde{r}_f^T + \tilde{V}_f^{(0)} + \big(\widetilde{\tilde{r}_f^T \boldsymbol{\omega}_f^{(0)}}\big) \big] \, \mathrm{d}m_f + \sum_{i=w,e} \int \Big\{ D_{uf}^T \Phi_{ufi}^T \tilde{\omega}_f^{(0)T} \big( \tilde{r}_{fi}^T + C_i^T \tilde{r}_i^T C_i \big) + D_{uf}^T \Phi_{ufi}^T \big[ \tilde{V}_f^{(0)} + \big(\widetilde{\tilde{r}_{fi}^T \boldsymbol{\omega}_f^{(0)}}\big) + \big(\widetilde{C_i^T \tilde{r}_i^T C_i \boldsymbol{\omega}_f^{(0)}}\big) \big] + D_{ui}^T \Phi_{ui}^T \Big[ \big(\widetilde{C_i \boldsymbol{\omega}_f^{(0)}}\big)^T \big( C_i \tilde{r}_{fi}^T + \tilde{r}_i^T C_i \big) + \big(\widetilde{C_i \boldsymbol{V}_f^{(0)}}\big) C_i + \big( C_i \tilde{r}_{fi}^T + \widetilde{\tilde{r}_i^T C_i} \big) \boldsymbol{\omega}_f^{(0)} C_i \Big] \Big\} \, \mathrm{d}m_i$$

$$A_{T\eta} = D_{uf}^T \int \Phi_{uf}^T \tilde{\omega}_f^{(0)T} \big( \Phi_{uf} D_{uf} + \tilde{r}_f \Phi_{\psi f} D_{\psi f} \big) \, \mathrm{d}m_f + \sum_{i=w,e} \int \Big[ D_{uf}^T \Phi_{ufi}^T \tilde{\omega}_f^{(0)T} C_i^T + D_{ui}^T \Phi_{ui}^T \big(\widetilde{C_i \boldsymbol{\omega}_f^{(0)}}\big)^T \Big] \times \big[ \tilde{r}_i^T C_i \Delta \Phi_{ufi} + C_i \Phi_{ufi} \big) D_{uf} + \Phi_{ui} D_{ui} + \big( \tilde{r}_i^T C_i + C_i \tilde{r}_{fi}^T \big) \Phi_{\psi fi} D_{\psi f} + \tilde{r}_i^T \Phi_{\psi i} D_{\psi i} \big] \, \mathrm{d}m_i \tag{26}$$

Finally, introducing the first-order state vector $\boldsymbol{x}^{(1)} = [\boldsymbol{R}_f^{(1)T} \;\; \boldsymbol{\theta}_f^{(1)T} \;\; \boldsymbol{\xi}^T \;\; \boldsymbol{p}_{Vf}^{(1)T} \;\; \boldsymbol{p}_{\omega f}^{(1)T} \boldsymbol{p}_\eta^T]^T$, recognizing that, for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\tilde{a}\boldsymbol{b} = -\tilde{b}\boldsymbol{a}$, and inserting Eqs. (13), (21–23), (25), and (26) into Eqs. (9), we can write the first-order state equations in the compact standard form

$$\dot{\boldsymbol{x}}^{(1)}(t) = A^*(t)\boldsymbol{x}^{(1)}(t) + B^* \boldsymbol{\mathcal{F}}^{(1)}(t) \tag{27}$$

where $A^*(t) = A^*(\boldsymbol{x}^{(0)}(t))$ is a $2(6+m) \times 2(6+m)$ matrix depending explicitly on the zero-order state vector, $B^*$ is a $2(6+m) \times (6+m)$ matrix with the top half equal to the null matrix and the bottom half equal to the identity matrix, and $\boldsymbol{\mathcal{F}}^{(1)}(t) = [\boldsymbol{F}^{(1)T}(t) \;\; \boldsymbol{M}^{(1)T}(t) \;\; \boldsymbol{Q}_\xi^T(t)]^T$ is the first-order generalized force vector. The matrix $A^*$ has a partitioned form with the $(6+m) \times (6+m)$ submatrices

$$A_{11}^* = \begin{bmatrix} 0 & C_{fV}^T & 0 \\ 0 & -E_{f\omega} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} C_f^{(0)T} & 0 & 0 \\ 0 & \big(E_f^{(0)}\big)^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} \big(M_\xi^{(0)}\big)^{-1} M_{\xi V} U_u$$

$$A_{12}^* = \begin{bmatrix} C_f^{(0)T} & 0 & 0 \\ 0 & \big(E_f^{(0)}\big)^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} \big(M_\xi^{(0)}\big)^{-1}$$

$$A_{21}^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_{T\xi} - K_\xi \end{bmatrix} - \begin{bmatrix} 0 & \tilde{p}_{Vf}^{(0)} & 0 \\ \tilde{p}_{Vf}^{(0)} & \tilde{p}_{\omega f}^{(0)} & 0 \\ A_{TV} & A_{T\omega} & A_{T\eta} - C_\eta \end{bmatrix} \big(M_\xi^{(0)}\big)^{-1} M_{\xi V} U_u$$

$$A_{22}^* = \begin{bmatrix} -\tilde{\omega}_f^{(0)} & 0 & 0 \\ -\tilde{V}_f^{(0)} & -\tilde{\omega}_f^{(0)} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \tilde{p}_{Vf}^{(0)} & 0 \\ \tilde{p}_{Vf}^{(0)} & \tilde{p}_{\omega f}^{(0)} & 0 \\ A_{TV} & A_{T\omega} & A_{T\eta} - C_\eta \end{bmatrix} \big(M_\xi^{(0)}\big)^{-1} \tag{28}$$

Clearly, if the aircraft maneuver depends explicitly on time, so does the matrix $A^*$, and the system is time varying. Otherwise, the system is time invariant. The generalized force vector $\mathcal{F}^{(1)}(t)$ contains contributions from the gravity, propulsion, aerodynamic, and control forces and is considered in more detail later in this paper.

## IV. Aerodynamic and Gravity Forces

Assuming that the angles of attack of the lift force and lateral force acting on the fuselage are small, so that $\sin\alpha_i \approx \alpha_i = \alpha_i^{(0)} + \alpha_i^{(1)}$ and $\cos\alpha_i \approx 1$ and $\sin\beta_i \approx \beta_i = \beta_i^{(0)} + \beta_i^{(1)}$ and $\cos\beta_i \approx 1 (i = f, w, e)$, which implies that both the zero-order parts and first-order parts are small, we conclude from Ref. 6 that the zero-order lift and lateral aerodynamic force densities on the fuselage can be written in the form

$$\boldsymbol{f}_{af}^{(0)} = \big[\alpha_f^{(0)} \quad 0 \quad -1\big]^T \ell_f^{(0)} + \big[-1 \quad 0 \quad -\alpha_f^{(0)}\big]^T d_f^{(0)}$$

$$\boldsymbol{f}_{sf}^{(0)} = \big[\beta_f^{(0)} \quad -1 \quad 0\big]^T s_f^{(0)} \tag{29}$$

in which the zero-order parts of the lift, drag, and lateral forces per unit area of fuselage are

$$\ell_f^{(0)} = q_f^{(0)} c_f C_{L\alpha f} \alpha_f^{(0)}, \qquad d_f^{(0)} = q_f^{(0)} c_f \big( C_{Df0} + k_f C_{L\alpha f}^2 \alpha_f^{(0)2} \big)$$

$$s_f^{(0)} = q_{sf}^{(0)} c_{sf} C_{s\beta f} \beta_f^{(0)} \tag{30}$$

where, for small $\alpha_f^{(0)}$ and $\beta_f^{(0)}$, we have $q_f^{(0)} \cong \frac{1}{2}\rho \bar{V}_{fx}^{(0)2}$ and $\alpha_f^{(0)} = \bar{V}_{fz}^{(0)}/\bar{V}_{fx}^{(0)}$ and $q_{sf}^{(0)} \cong \frac{1}{2}\rho \bar{V}_{fx}^{(0)2}$ and $\beta_f^{(0)} = \bar{V}_{fy}^{(0)}/\bar{V}_{fx}^{(0)}$, in which $\bar{V}_{fx}^{(0)}$, $\bar{V}_{fy}^{(0)}$, and $\bar{V}_{fz}^{(0)}$ are the body-axes components of $\bar{\boldsymbol{V}}_f^{(0)} = [0 \;\; 0 \;\; I \;\; \tilde{r}_f^T] \times$ block-diag $[I \;\; (M_{rr}^{(0)})^{-1}]\boldsymbol{x}^{(0)}$, where due consideration was given to Eqs. (5) and (8).

Similarly, for the wing and empennage, we have

$$\boldsymbol{f}_{ai}^{(0)} = \big[0 \quad \alpha_i^{(0)} \quad -1\big]^T \ell_i^{(0)} + \big[0 \quad -1 \quad -\alpha_i^{(0)}\big]^T d_i^{(0)}, \qquad i = w, e$$

$$\boldsymbol{f}_{se}^{(0)} = \big[0 \quad \beta_e^{(0)} \quad -1\big]^T s_e^{(0)} \tag{31}$$

in which

$$\ell_w^{(0)} = q_w^{(0)} \big( c_w C_{L\alpha w} \alpha_w^{(0)} + c_{\delta a} C_{L\delta a} \delta_a^{(0)} \big)$$

$$\ell_e^{(0)} = q_e^{(0)} \big( c_e C_{L\alpha e} \alpha_e^{(0)} + c_{\delta e} C_{L\delta e} \delta_e^{(0)} \big)$$

$$d_i^{(0)} = q_i^{(0)} c_i \big( C_{Di0} + k_i C_{L\alpha i}^2 \alpha_i^{(0)2} \big), \qquad i = w, e$$

$$s_e^{(0)} = q_{se}^{(0)} \big( c_{se} C_{s\beta e} \beta_e^{(0)} + c_{\delta r} C_{s\delta r} \delta_r^{(0)} \big) \tag{32}$$

where $q_i^{(0)} \cong \frac{1}{2}\rho \bar{V}_{iy}^{(0)2}$; $\alpha_i^{(0)} = \bar{V}_{iz}^{(0)}/\bar{V}_{iy}^{(0)}$, $i = w, e$; $q_{se}^{(0)} \cong \frac{1}{2}\rho \bar{V}_{ey}^{(0)2}$; and $\beta^{(0)} = \bar{V}_z^{(0)}/\bar{V}_y^{(0)}$, in which $\bar{V}_{iy}^{(0)}$ and $\bar{V}_{iz}^{(0)}$ are components of $\bar{\boldsymbol{V}}_i^{(0)}$, $i = w, e$.

The zero-order parts of the gravity force densities are simply

$$\boldsymbol{f}_{gf}^{(0)} = C_f^{(0)} [0 \quad 0 \quad \rho_f g]^T, \qquad \boldsymbol{f}_{gi}^{(0)} = C_i C_f^{(0)} [0 \quad 0 \quad \rho_i g]^T \qquad i = w, e \tag{33}$$

On inserting Eqs. (29), (31), and (33) into Eqs. (2), we obtain $\boldsymbol{F}^{(0)}$ and $\boldsymbol{M}^{(0)}$, which are used, in turn, to determine the nonlinear function $\boldsymbol{f}_2$ and the coefficient matrix $B^{(0)}$ defining the zero-order state equations, Eq. (6).

Following the same pattern as for the zero-order quantities and using results from Ref. 6, the first-order lift and lateral aerodynamic force densities acting on the fuselage can be shown to have the form

$$\boldsymbol{f}_{af}^{(1)} = F_{af}^{(0)}\boldsymbol{x}^{(1)}, \qquad \boldsymbol{f}_{sf}^{(1)} = F_{sf}^{(0)}\boldsymbol{x}^{(1)} \tag{34}$$

in which $F_{af}^{(0)}$ is a matrix depending on $\bar{V}_{fx}^{(0)}$, $\ell_f^{(0)}$, $d_f^{(0)}$, and $\alpha_f^{(0)}$ and $F_{sf}^{(0)}$ is a matrix depending on $\bar{V}_{fx}^{(0)}$, $s_f^{(0)}$, and $\beta_f^{(0)}$. Similarly,

$$\boldsymbol{f}_{aw}^{(1)} = F_{aw}^{(0)}\boldsymbol{x}^{(1)} + F_{w\psi}^{(0)}\boldsymbol{x}^{(1)} + \boldsymbol{f}_{\delta a}^{(0)}\delta_a^{(1)} \tag{35}$$

in which the matrix $F_{aw}^{(0)}$ depends on $\bar{V}_{wy}^{(0)}$, $\ell_w^{(0)}$, $d_w^{(0)}$, and $\alpha_w^{(0)}$, the matrix $F_{w\psi}^{(0)}$ depends on $\ell_w^{(0)}$, $d_w^{(0)}$, and $\alpha_w^{(0)}$, and the vector $\boldsymbol{f}_{\delta a}^{(0)}$ depends on $\alpha_w^{(0)}$, and

$$\boldsymbol{f}_{ae}^{(1)} = F_{ae}^{(0)}\boldsymbol{x}^{(1)} + F_{e\psi}^{(0)}\boldsymbol{x}^{(1)} + \boldsymbol{f}_{\delta e}^{(0)}\delta_e^{(1)} \tag{36}$$

where $F_{ae}^{(0)}$ depends on $\bar{V}_{ey}^{(0)}$, $\ell_e^{(0)}$, $d_e^{(0)}$, and $\alpha_e^{(0)}$; $F_{e\psi}^{(0)}$ depends on $\ell_e^{(0)}$, $d_e^{(0)}$, and $\alpha_e^{(0)}$; and $\boldsymbol{f}_{\delta e}^{(0)}$ depends on $\alpha_e^{(0)}$. Moreover,

$$\boldsymbol{f}_{se}^{(1)} = F_{se}^{(0)}\boldsymbol{x}^{(1)} + F_{se\psi}^{(0)}\boldsymbol{x}^{(1)} + \boldsymbol{f}_{\delta se}^{(0)}\delta_r^{(1)} \tag{37}$$

in which $F_{se}^{(0)}$ depends on $\bar{V}_{ey}^{(0)}$, $s_e^{(0)}$, and $\beta_e^{(0)}$, and $F_{se\psi}^{(0)}$ and $\boldsymbol{f}_{\delta se}^{(0)}$ depend on $\beta_f^{(0)}$. Note that complete expressions for $F_{af}^{(0)}$, $F_{sf}^{(0)}$, $F_{aw}^{(0)}, \ldots, F_{se\psi}^{(0)}$, and $\boldsymbol{f}_{\delta se}^{(0)}$ are given in an earlier version of this paper (Ref. 9) and they were omitted here for brevity.

Using the first of Eqs. (10), the first-order gravity force densities are given by

$$\boldsymbol{f}_{gf}^{(1)} = F_{gf}^{(0)}\boldsymbol{x}^{(1)}, \qquad \boldsymbol{f}_{gi}^{(1)} = F_{gi}^{(0)}\boldsymbol{x}^{(1)}, \qquad i = w, e \tag{38}$$

where

$$F_{gf}^{(0)} = \rho_f g \bar{C}_f^{(0)}[0 \quad I \quad 0 \quad 0 \quad 0 \quad 0]$$

$$F_{gi}^{(0)} = \rho_i g C_i \bar{C}_f^{(0)}[0 \quad I \quad 0 \quad 0 \quad 0 \quad 0], \qquad i = w, e \tag{39}$$

in which $\bar{C}_f^{(0)}$ is a matrix depending on $C_{f\psi}^{(0)}$, $C_{f\theta}^{(0)}$, and $C_{f\phi}^{(0)}$, where the latter are defined by Eqs. (11).

## V. Control Forces for the Extended Perturbation Problem

The perturbation approach used here considers the way in which flexible aircraft are flown to separate the problem into one for the large rigid-body motions on a certain trajectory and another for small deviations in the rigid-body motions and the elastic deformations. The first represents a quasi-rigid flight dynamics problem and the second is referred to as an extended perturbation problem. The extended perturbation problem receives input from the flight dynamics problem, which can create difficulties when the aircraft executes a time-dependent maneuver.

The control design for the quasi-rigid flight dynamics problem is discussed in the next section. In this section, we concentrate on the controls for the extended perturbation problem.

We assume that the controls for the extended perturbation problem are carried out by the engine throttle, aileron, elevator, and rudder, as well as point actuators on the wing and empennage; the role of the latter is to suppress vibration. To demonstrate the idea, we assume that the point actuators are placed as follows: two actuators each on the right half-wing, left half-wing, right half-horizontal stabilizer, left half-horizontal stabilizer, and vertical stabilizer. Then, we denote the first-order control vector by $\boldsymbol{u}^{(1)}$ and divide it into two parts, one part $\boldsymbol{u}_1^{(1)} = [F_E^{(1)} \quad \delta_a^{(1)} \quad \delta_e^{(1)} \quad \delta_r^{(1)}]^T$ due to the standard aircraft controls and another part $\boldsymbol{u}_2^{(1)}$ due to controls for suppressing the elastic vibration, so that $\boldsymbol{u}^{(1)} = [\boldsymbol{u}_1^{(1)T} \quad \boldsymbol{u}_2^{(1)T}]^T$. We further assume that the control law for the first part is given by

$$\boldsymbol{u}_1^{(1)} = -G\boldsymbol{x}^{(1)} \tag{40}$$

in which the control gain matrix $G$ is determined by the LQR method.[8]

To determine the contribution of $\boldsymbol{u}_2^{(1)}$ to the first-order force vector, Eqs. (14), we use the direct feedback control method[8] with collocated sensors and actuators, so that the control law is assumed to have the generic form

$$F_i(x_{ij}, t) = -k_j u_i(x_{ij}, t) - c_j \dot{u}_i(x_{ij}, t)$$
$$i = w, \qquad j = 1, 2, 3, 4; \qquad i = e, \qquad j = 5, 6, \ldots, 10 \tag{41}$$

where $k_j$ and $c_j$ are constant gain coefficients and $u_i(x_{ij}, t)$ and $\dot{u}_i(x_{ij}, t)$ are elastic displacement and velocity at the locations $x_{ij}$ of the respective actuators. Note that this notation is no accident because the first term in Eqs. (41) plays the role of control-induced stiffness forces and the second term can be regarded as control-induced damping forces. With a view to the developments to follow, we recognize that the point actuators act in the local $z_i$ direction and represent them as distributed force vectors by writing

$$\boldsymbol{f}_i(x_i, t) = [0 \quad 0 \quad F_i(x_{ij}, t)]^T \delta(x_i - x_{ij})$$
$$i = w, \qquad j = 1, 2, 3, 4; \qquad i = e, \qquad j = 5, 6, \ldots, 10 \tag{42}$$

where $\delta(x_i - x_{ij})$ are spatial Dirac delta functions. But, because the state equations, Eqs. (9), are in terms of aircraft generalized coordinates, we must express the control law in terms of the same coordinates. To this end, for any point in position $x_i$ on a given aircraft flexible component, we write

$$u_i(x_i, t) = \boldsymbol{\phi}_{ui}^T(x_i)\boldsymbol{q}_{ui}(t) = \boldsymbol{\phi}_{ui}^T(x_i) D_{ui}\boldsymbol{\xi}(t), \qquad i = w, e \tag{43}$$

in which $\boldsymbol{\phi}_{ui}(x_i)$ is a vector of component shape functions and $\boldsymbol{q}_{ui}(t)$ is a corresponding vector of generalized coordinates. Moreover, $\boldsymbol{\phi}_{ui}^T(x_i) D_{ui}$ can be identified as a vector of aircraft shape functions.[7] Then, inserting Eqs. (43) into Eqs. (42), we obtain the distributed control force vectors

$$\boldsymbol{f}_{ci}^{(1)}(x_i, t) = \begin{bmatrix} 0 \\ 0 \\ F_i(x_{ij}, t) \end{bmatrix} \delta(x_i - x_{ij}) = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\times\, \boldsymbol{\phi}_{ui}^T(x_{ij}) D_{ui}\delta(x_i - x_{ij})(k_j\boldsymbol{\xi} + c_j\boldsymbol{\eta}) = F_{ci}^{(0)}(x_i)\boldsymbol{x}^{(1)} \tag{44}$$

in which, with due consideration of Eqs. (21) and (22),

$$F_{ci}^{(0)}(x_i) = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \boldsymbol{\phi}_{ui}^T(x_{ij}) D_{ui}\delta(x_i - x_{ij})$$
$$\times \begin{bmatrix} 0 & 0 & k_j I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_j I \end{bmatrix} \begin{bmatrix} I & 0 \\ -\left(M_\xi^{(0)}\right)^{-1} M_{\xi V} U & \left(M_\xi^{(0)}\right)^{-1} \end{bmatrix} \tag{45}$$

Finally, introducing Eq. (44) into Eqs. (14), we can write the contribution of the point actuators to the first-order force vector in the form

$$\mathcal{F}_c^{(1)} = -H_c(t)\boldsymbol{x}^{(1)} \tag{46}$$

where the control gain matrix for direct feedback is as follows:

$$H_c = -\begin{bmatrix} \displaystyle\sum_{i=w,e} C_i^T \int_{D_i} F_{ci}^{(0)}(x_i)\,\mathrm{d}D_i \\ \displaystyle\sum_{i=w,e} \int_{D_i} \left(\tilde{r}_{fi} C_i^T + C_i^T \tilde{r}_i\right) F_{ci}^{(0)}(x_i)\,\mathrm{d}D_i \\ \displaystyle\sum_{i=w,e} \int_{D_i} \Big[\left(\tilde{r}_i^T C_i \Delta\boldsymbol{\Phi}_{ufi} + C_i \boldsymbol{\Phi}_{ufi}\right) D_{uf} + \Phi_{ui} D_{ui} \\ \quad + \left(\tilde{r}_i^T C_i + C_i \tilde{r}_{fi}^T\right)\Phi_{\psi fi} D_{\psi f} + \tilde{r}_i^T \Phi_{\psi i} D_{\psi i}\Big] F_{ci}^{(0)}(x_i)\,\mathrm{d}D_i \end{bmatrix} \tag{47}$$

Finally, we insert Eqs. (34–38) and (46) into Eqs. (14) and express the contributions from aerodynamics, gravity, and controls to the first-order generalized force vector in the form

$$\mathcal{F}^{(1)}(t) = \begin{bmatrix} \boldsymbol{F}^{(1)}(t) \\ \text{-----} \\ \boldsymbol{M}^{(1)}(t) \\ \text{-----} \\ \boldsymbol{Q}_\xi(t) \end{bmatrix} = \begin{bmatrix} H_F(t) \\ \text{-----} \\ H_M(t) \\ \text{-----} \\ H_Q(t) \end{bmatrix} \boldsymbol{x}^{(1)}(t) - H_c(t)\boldsymbol{x}^{(1)}(t)$$
$$+ \begin{bmatrix} B_F^{**}(t) \\ \text{-----} \\ B_M^{**}(t) \\ \text{-----} \\ B_Q^{**}(t) \end{bmatrix} \boldsymbol{u}_1^{(1)}(t) = [H(t) - H_c(t)]\boldsymbol{x}^{(1)}(t) + B^{**}(t)\boldsymbol{u}_1^{(1)}(t) \tag{48}$$

where

$$H_F = \int_{D_f} \left(F_{af}^{(0)} + F_{sf}^{(0)} + F_{gf}^{(0)}\right) \mathrm{d}D_f + C_w^T \int_{D_w} \left(F_{aw}^{(0)} + F_{w\psi}^{(0)} + F_{gw}^{(0)}\right) \mathrm{d}D_w + C_e^T \int_{D_e} \left(F_{ae}^{(0)} + F_{e\psi}^{(0)} + F_{se}^{(0)} + F_{se\psi}^{(0)} + F_{ge}^{(0)}\right) \mathrm{d}D_e$$

$$H_M = \int_{D_f} \left\{\tilde{r}_f\left(F_{af}^{(0)} + F_{sf}^{(0)} + F_{gf}^{(0)}\right) + \left[\tilde{f}_f^{(0)T} + \tilde{F}_E^{(0)}\delta(\boldsymbol{r}_f - \boldsymbol{r}_E)\right] \times \Phi_{uf} D_{uf}[0 \quad 0 \quad I \quad 0 \quad 0 \quad 0]\right\} \mathrm{d}D_f$$
$$+ \sum_{i=w,e} C_i^T \int_{D_i} \left\{\left(\tilde{r}_{fi} C_i^T + C_i^T \tilde{r}_i\right)\left(F_{ai}^{(0)} + F_{i\psi}^{(0)} + F_{gi}^{(0)}\right) + \left[\left(\widetilde{C_i^T \boldsymbol{f}_i^{(0)}}\right)^T \Phi_{ufi} D_{uf} + C_i^T \tilde{f}_i^{(0)T} \Phi_{ui} D_{ui}\right] \times [0 \quad 0 \quad I \quad 0 \quad 0 \quad 0]\right\} \mathrm{d}D_i$$

$$H_Q = \int_{D_f} \left(\Phi_{uf} D_{uf} + \tilde{r}_f^T \Phi_{\psi f} D_{\psi f}\right)^T \left(F_{af}^{(0)} + F_{sf}^{(0)} + F_{gf}^{(0)}\right) \mathrm{d}D_f$$
$$+ \int_{D_w} \left[\left(\tilde{r}_w^T C_w \Delta\boldsymbol{\Phi}_{ufw} + C_w \Phi_{ufw}\right) D_{uf} + \Phi_{uw} D_{uw} + \left(\tilde{r}_w^T C_w + C_w \tilde{r}_{fw}^T\right) \Phi_{\psi fw} D_{\psi f} + \tilde{r}_w^T \Phi_{\psi w} D_{\psi w}\right]^T \times \left(F_{aw}^{(0)} + F_{w\psi}^{(0)} + F_{gw}^{(0)}\right) \mathrm{d}D_w$$
$$+ \int_{D_e} \left[\left(\tilde{r}_e^T C_e \Delta\boldsymbol{\Phi}_{ufe} + C_e \Phi_{ufe}\right) \times D_{uf} + \Phi_{ue} D_{ue} + \left(\tilde{r}_e^T C_e + C_e \tilde{r}_{fe}^T\right) \Phi_{\psi fe} D_{\psi f} + \tilde{r}_e^T \Phi_{\psi e} D_{\psi e}\right]^T \left(F_{ae}^{(0)} + F_{e\psi}^{(0)} + F_{se}^{(0)} + F_{se\psi}^{(0)} + F_{ge}^{(0)}\right) \mathrm{d}D_e$$

$$B_F^{**} = \left[\boldsymbol{e}_1 \quad C_w^T \int_{D_w} \boldsymbol{f}_{\delta a}^{(0)} \mathrm{d}D_w \quad C_e^T \int_{D_e} \boldsymbol{f}_{\delta e}^{(0)} \mathrm{d}D_e \quad C_e^T \int_{D_e} \boldsymbol{f}_{\delta se}^{(0)} \mathrm{d}D_e\right]$$

$$B_M^{**} = \left[\tilde{r}_f(\boldsymbol{r}_E)\boldsymbol{e}_1 \quad \int_{D_w} \left(\tilde{r}_{fw} C_w^T + C_w^T \tilde{r}_w\right) \boldsymbol{f}_{\delta a}^{(0)} \mathrm{d}D_w \quad \int_{D_e} \left(\tilde{r}_{fe} C_e^T + C_e^T \tilde{r}_e\right) \boldsymbol{f}_{\delta e}^{(0)} \mathrm{d}D_e \quad \int_{D_e} \left(\tilde{r}_{fe} C_e^T + C_e^T \tilde{r}_e\right) \boldsymbol{f}_{\delta se}^{(0)} \mathrm{d}D_e\right]$$

$$B_Q^{**} = \Bigg[\left[\Phi_{uf}(\boldsymbol{r}_E) D_{uf} + \tilde{r}_f^T(\boldsymbol{r}_E)\Phi_{\psi f}(\boldsymbol{r}_E) D_{\psi f}\right]\boldsymbol{e}_1$$
$$\int_{D_w} \left[\left(\tilde{r}_w^T C_w \Delta\boldsymbol{\Phi}_{ufw} + C_w \Phi_{ufw}\right) D_{uf} + \Phi_{uw} D_{uw} + \left(\tilde{r}_w^T C_w + C_w \tilde{r}_{fw}^T\right) \Phi_{\psi fw} D_{\psi f} + \tilde{r}_w^T \Phi_{\psi w} D_{\psi w}\right]^T \boldsymbol{f}_{\delta a}^{(0)} \mathrm{d}D_w$$
$$\int_{D_e} \left[\left(\tilde{r}_e^T C_e \Delta\boldsymbol{\Phi}_{ufe} + C_e \Phi_{ufe}\right) D_{uf} + \Phi_{ue} D_{ue} + \left(\tilde{r}_e^T C_e + C_e \tilde{r}_{fe}^T\right) \Phi_{\psi fe} D_{\psi f} + \tilde{r}_e^T \Phi_{\psi e} D_{\psi e}\right]^T \boldsymbol{f}_{\delta e}^{(0)} \mathrm{d}D_e$$
$$\int_{D_e} \left[\left(\tilde{r}_e^T C_e \Delta\boldsymbol{\Phi}_{ufe} + C_e \Phi_{ufe}\right) D_{uf} + \Phi_{ue} D_{ue} + \left(\tilde{r}_e^T C_e + C_e \tilde{r}_{fe}^T\right) \Phi_{\psi fe} D_{\psi f} + \tilde{r}_e^T \Phi_{\psi e} D_{\psi e}\right]^T \boldsymbol{f}_{\delta se}^{(0)} \mathrm{d}D_e \Bigg] \tag{49}$$

In Eq. (48), $H(t)$ is a matrix of coefficients for the aerodynamic, distributed controls and gravity forces, $B^{**}(t)$ is a matrix of coefficients for the control forces, and $\boldsymbol{u}_1^{(1)}(t) = [F_E^{(1)} \quad \delta_a^{(1)} \quad \delta_e^{(1)} \quad \delta_r^{(1)}]^T$ is part of the first-order control vector. (See next section.) It follows that we can rewrite the state equation, Eq. (27), in the standard closed-loop from

$$\dot{\boldsymbol{x}}^{(1)}(t) = A(t)\boldsymbol{x}^{(1)}(t) \tag{50}$$

in which

$$A(t) = A^*(t) + B^*\left[H(t) - B^{**}G - H_c\right] \tag{51}$$

is the closed-loop system matrix, in which the first term represents the open-loop system matrix and the second the control coefficient matrix.

## VI. Control in Discrete Time

The zero-order state equations are given by Eq. (6) and the first-order state equations by Eq. (50). When the zero-order state vector $\boldsymbol{x}^{(0)}$ is constant, Eq. (6) represents a set of nonlinear algebraic equations to be solved for the remaining quantities, all constant.[7] Then, the system matrix $A$ appearing in Eq. (50) is also constant, and the solution has an exponential form.[7] The situation is significantly more involved when $\boldsymbol{x}^{(0)}(t)$ represents a time-dependent maneuver because in this case Eq. (50) represents a set of time-dependent equations, which makes its solution very difficult. We consider hybrid control design, that is, we determine $G$ by the LQR method and $H_c$ by direct feedback control. In the case of time-varying systems, to determine the matrix $G(t)$, it is necessary to solve a transient Riccati equation (see Ref. 8), a nonlinear matrix differential equation. For high-order systems of the type exemplified by the flexible aircraft under consideration, a solution may not even be feasible. In view of this, it seems natural to seek a solution in discrete time.[8] This requires a sequence of solutions of steady-state Riccati equations, which are only algebraic. The idea of computing in discrete time is not as esoteric as it may sound when we recall that this is the way in which solutions are processed on digital computers.

In discrete-time processing of solutions, the continuous time $t$ is replaced by the sequence of discrete times $t_k$, $k = 0, 1, 2, \ldots$, known as sampling times. In general, the sampling is done periodically, so that $t_k = kT$, $k = 0, 1, 2, \ldots$, where $T$ is a constant called the sampling period, more commonly known as time step. For convenience, we denote the sampling times $t_k = kT$ simply by $k$. Because there are no time derivatives in discrete-time systems, we must approximate differential equations by difference equations. The accuracy of discrete-time solutions is ensured by choosing the time step $T$ sufficiently small.

Next, we rewrite the zero-order equation (6), in the form

$$\dot{\boldsymbol{x}}^{(0)}(t) = \boldsymbol{f}(t) + B^{(0)}(t)\boldsymbol{u}^{(0)}(t) \tag{52}$$

where we introduced the notation $\boldsymbol{f}_1(\boldsymbol{x}^{(0)}(t)) + \boldsymbol{f}_2(\boldsymbol{x}^{(0)}(t)) = \boldsymbol{f}(\boldsymbol{x}^{(0)}(t)) = \boldsymbol{f}(t)$. Then, approximating the time derivative $\dot{\boldsymbol{x}}^{(0)}(t)$ by $(\boldsymbol{x}^{(0)}(k+1) - \boldsymbol{x}^{(0)}(k))/T$, we discretize Eq. (52) in time and write

$$\boldsymbol{x}^{(0)}(k+1) = \boldsymbol{x}^{(0)}(k) + T\boldsymbol{f}(k) + TB^{(0)}(k)\boldsymbol{u}^{(0)}(k), \qquad k = 0, 1, 2, \ldots \tag{53}$$

Equations (53) represent a sequence of nonlinear algebraic equations in which all quantities are treated as constant over the periods from $kT$ to $(k+1)T$. Hence, given the zero-order state vector $\boldsymbol{x}^{(0)}(t)$ describing a certain aircraft maneuver, we sample it at the sampling times $t_k$, generate the sequence $\boldsymbol{x}^{(0)}(0), \boldsymbol{x}^{(0)}(1), \boldsymbol{x}^{(0)}(2), \ldots$, solve Eqs. (53), and derive sequences for the needed unknowns. The pertinent sequences are entered as inputs into the first-order equations. Note that all the computations required for producing these sequences can be carried out offline and the results introduced into the first-order equations as the need arises.

Finally, after providing for an external force, we obtain the first-order closed-loop equation

$$\dot{\boldsymbol{x}}^{(1)}(t) = A(t)\boldsymbol{x}^{(1)}(t) + B^*(t)\boldsymbol{F}_{\text{ext}}(t) \tag{54}$$

where $\boldsymbol{F}_{\text{ext}}(t)$ is a vector of external generalized forces. The discrete-time counterpart of Eq. (54) is simply

$$\boldsymbol{x}^{(1)}(k+1) = [I + TA(k)]\boldsymbol{x}^{(1)}(k) + TB^*(k)\boldsymbol{F}_{\text{ext}}(k) \qquad k = 0, 1, 2, \ldots \tag{55}$$

and we note that $A(k)$ includes inputs depending on $k$ from the zero-order problem, Eqs. (53). For any given maneuver sequence $\boldsymbol{x}^{(0)}(k)$ and any excitation sequence $\boldsymbol{F}_{\text{ext}}(k)$, Eqs. (55) can be solved recursively to obtain a sequence of state vectors $\boldsymbol{x}^{(1)}(k)$, $k = 0, 1, 2, \ldots$, describing the response of the system in discrete time. We recall that the components of the state vector include displacements and momenta, where the latter can be transformed into velocities. Every one of these quantities can be plotted vs $k$ to generate time simulations. In this regard, we recall that $k$ really stands for $kT$. These plots consist of discrete dots. For small $T$, however, that is, for high resolution, the plots appear as continuous.

## VII. Numerical Example

A numerical example requires geometric, structural, and aerodynamic data. Such data pertaining to an actual aircraft have been provided by an executive jet manufacturer and can be found in Ref. 6. As an example, we consider a pitch maneuver for which the solution of the zero-order problem depends on time and, consequently, the first-order equations describing the extended perturbation problem are time varying. For a pitch maneuver, the zero-order yaw and roll angles and the second component of $\boldsymbol{R}_f^{(0)}$ are zero, $\psi^{(0)} = \phi^{(0)} = 0$ and $R_{fy}^{(0)} = 0$. The pitch angle is assumed to have the form

$$\theta^{(0)}(t) = \hat{\theta}(t) + \theta_D(t) \tag{56}$$

where $\theta_D(t)$ is the desired pitch angle throughout the maneuver, so that $\theta^{(0)}(t)$ approaches $\theta_D(t)$ as $\hat{\theta}(t)$ approaches zero. Similarly, the first and third components of $\boldsymbol{R}_f^{(0)}$ are assumed to have the form

$$R_X^{(0)}(t) = \hat{R}_X(t) + R_{XD}(t), \qquad R_Z^{(0)}(t) = \hat{R}_Z(t) + R_{ZD}(t) \tag{57}$$

where $R_{XD}(t)$ and $R_{ZD}(t)$ are the solutions of the differential equations $\dot{R}_{XD} = V^{(0)}\cos\theta_D$ and $\dot{R}_{ZD} = -V^{(0)}\sin\theta_D$, in which $V^{(0)}$ is the aircraft forward velocity, so that $\tan\theta_D = -\mathrm{d}R_{ZD}/\mathrm{d}R_{XD}$. On the other hand, the zero-order velocity vectors are

$$\boldsymbol{V}_f^{(0)} = \begin{bmatrix} V_{fx}^{(0)} & 0 & V_{fz}^{(0)} \end{bmatrix}^T, \qquad \boldsymbol{\omega}_f^{(0)} = \begin{bmatrix} 0 & \omega_{fy}^{(0)} & 0 \end{bmatrix}^T \tag{58}$$

Moreover, from Eqs. (5), the momentum vectors are

$$\begin{aligned} \boldsymbol{p}_{Vf}^{(0)} &= m\begin{bmatrix} V_{fx}^{(0)} & 0 & V_{fz}^{(0)} \end{bmatrix}^T + \tilde{S}^{(0)T}\begin{bmatrix} 0 & \omega_{fy}^{(0)} & 0 \end{bmatrix}^T \\ \boldsymbol{p}_{\omega f}^{(0)} &= \tilde{S}^{(0)}\begin{bmatrix} V_{fx}^{(0)} & 0 & V_{fz}^{(0)} \end{bmatrix}^T + J^{(0)}\begin{bmatrix} 0 & \omega_{fy}^{(0)} & 0 \end{bmatrix}^T \end{aligned} \tag{59}$$

Using Eqs. (56–58), the kinematical relations given by the first two of Eqs. (1) reduce to

$$\begin{aligned} \dot{\hat{R}}_X &= \cos(\hat{\theta} + \theta_D)V_{fx}^{(0)} + \sin(\hat{\theta} + \theta_D)V_{fz}^{(0)} - \dot{R}_{XD} \\ \dot{\hat{R}}_Z &= -\sin(\hat{\theta} + \theta_D)V_{fx}^{(0)} + \cos(\hat{\theta} + \theta_D)V_{fD}^{(0)} - \dot{R}_{ZD} \\ \dot{\hat{\theta}} &= \omega_{fy} - \dot{\theta}_D \end{aligned} \tag{60}$$

On the other hand, using the same relations and observing that $\delta_r^{(0)} = 0$, the third and fourth of Eqs. (1) reduce to three scalar equations. Hence, the three control variables, namely, the engine thrust $F_E^{(0)}$, the aileron angle $\delta_a^{(0)}$, and the elevator angle $\delta_e^{(0)}$ must be chosen such that $\hat{R}_X$, $\hat{R}_Z$, $\hat{\theta}$, and $\omega_{fy}^{(0)}$ remain as close to zero as possible, whereas $V_{fx}^{(0)}$ and $V_{fz}^{(0)}$ remain constant.

The zero-order state equations, Eq. (52), are nonlinear and time varying. For practical reasons, we must seek a solution in discrete time, which implies use of Eqs. (53). This further implies that we must first linearize and then discretize the nonlinear vector function $\boldsymbol{f}(t)$ in time. To this end, we write

$$\boldsymbol{f}(t) = \boldsymbol{f}\big(\boldsymbol{x}^{(0)}(t)\big) = A^{(0)}(t)\boldsymbol{x}^{(0)}(t) \tag{61}$$

where

$$A^{(0)} = \left[a_{ij}^{(0)}\right] = \left[\frac{\partial f_i^{(0)}}{\partial x_j^{(0)}}\right] \tag{62}$$

is a time-varying matrix. Hence, the discrete-time counterpart of Eq. (61) is simply

$$\boldsymbol{f}(k) = A^{(0)}(k)\boldsymbol{x}^{(0)}(k), \qquad k = 0, 1, 2, \ldots \tag{63}$$

Inserting Eqs. (63) into Eqs. (53), we obtain

$$\boldsymbol{x}^{(0)}(k+1) = \left[I + TA^{(0)}(k)\right]\boldsymbol{x}^{(0)}(k) + TB^{(0)}(k)\boldsymbol{u}^{(0)}(k), \qquad k = 0, 1, 2, \ldots \tag{64}$$

Equations (64) can be solved sequentially by inverse dynamics, which amounts to assuming a sequence of zero-order state vectors $\boldsymbol{x}^{(0)}(k)$ and deriving a sequence of control vectors $\boldsymbol{u}^{(0)}(k)$, $k = 0, 1, 2, \ldots$. As an alternative, it is perhaps simpler to postulate a maneuver geometry and derive an optimal control vector in the form

$$\boldsymbol{u}^{(0)}(k) = -G^{(0)}(k)\boldsymbol{x}^{(0)}(k), \qquad k = 0, 1, 2, \ldots \tag{65}$$

where the control gain matrices are obtained for every $k$ by the LQR method, which amounts to solving a sequence of steady-state Riccati equations. This requires that the sequence closed-loop matrices $A^{(0)}(k) - B^{(0)}(k)G^{(0)}(k)$, $k = 0, 1, 2, \ldots$, have stable eigenvalues.

We assume that $\theta_D(t)$ has the functional dependence

$$\begin{aligned} \theta_D(t) = -0.06\{&[u(t) - u(t - T_m)](1 - \cos 2\pi t/T_m) \\ &+ [u(t - 2T_m) - u(t - 3T_m)][1 - \cos 2\pi(t - 2T_m)/T_m]\} \end{aligned} \tag{66}$$

in which $u(t-a)$ is the unit step function initiated at $t = a$. We note that the time derivative of $\theta_D$ vanishes at both the beginning and the end of the maneuver, which ensures a smooth transition from the steady rectilinear flight before the maneuver to a steady rectilinear flight after it. Figure 1 shows $\theta_D(t)$ vs $t$ and Fig. 2 shows $R_{ZD}(t)$ vs $R_{XD}(t)$, which give a fair idea of the nature of the maneuver. Note that Figs. 1 and 2 are for the maneuver duration $T_m = 8$ s.

We assume that, before the pitch maneuver, the aircraft is on a steady level flight at an altitude of 25,000 ft and with a forward velocity of $V^{(0)} = 416.67$ ft/s. The control parameters are

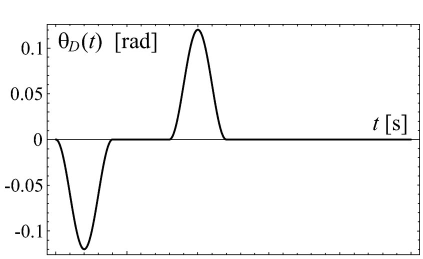


**Fig. 1 Desired pitch angle.**

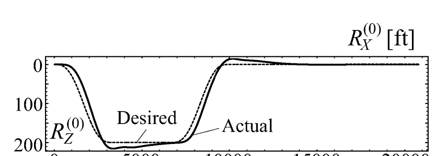


**Fig. 2 Flight trajectory in $X_f Y_f$ plane.**

$F_E^{(0)} = 433.09$ lb, $\delta_a^{(0)} = 0$, and $\delta_e^{(0)} = -0.3178$ rad, the angle of attack is $\alpha^{(0)} = 0.0680$ rad, and the velocity components are $V_{fx}^{(0)} = V^{(0)} \cos\alpha^{(0)} = 415.70$ ft/s and $V_{fz}^{(0)} = V^{(0)} \sin\alpha^{(0)} = 28.32$ ft/s.

The weighting matrices in the performance measure for the LQR method are chosen as

$$Q = \mathrm{diag}[500 \quad 500 \quad 0 \quad 0 \quad 0 \quad 0], \qquad R = \mathrm{diag}[1 \quad 10^8 \quad 10^{10}] \tag{67}$$

The resulting control gain matrices $G^{(0)}(k)$, when inserted into Eqs. (65), yield the control vectors sequence $\boldsymbol{u}^{(0)}(k)$. The corresponding components $F_E^{(0)}(k), \delta_a^{(0)}(k)$, and $\delta_e^{(0)}(k)$ are shown in Figs. 3–5. Similarly, the resulting $R_Z^{(0)}(k)$ vs $R_X^{(0)}(k)$ is displayed as a solid curve in Fig. 2.

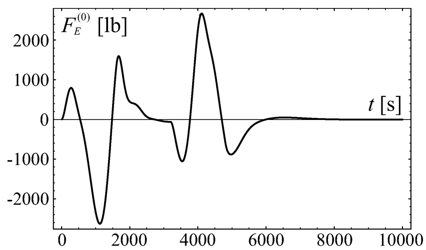


**Fig. 3 Zero-order engine thrust.**

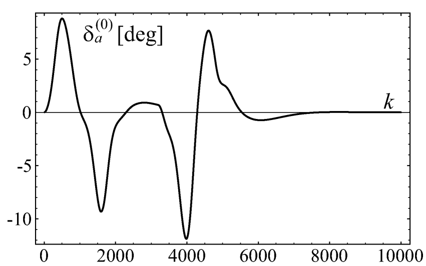


**Fig. 4 Zero-order aileron angle.**

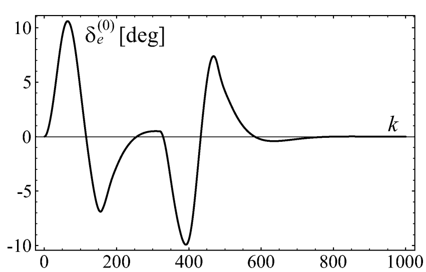


**Fig. 5 Zero-order elevator angle.**

The structural model of the flexible aircraft is shown in Fig. 6; it is the same as that used in Ref. 7. For the first-order problem, we model the flexibility by means of the first two aircraft shape functions; they are displayed in Fig. 3 of Ref. 7. We assume that, before the maneuver, the aircraft maintains the first-order forward velocity $V^{(1)} = -5$ in./s, so that $\boldsymbol{V}_f^{(1)} = C_f^{(0)}[-5 \ 0 \ 0] - C_f^{(1)} V_f^{(0)}$ and a zero angular velocity, $\boldsymbol{\omega}_f^{(1)} = \boldsymbol{0}$. In the steady flight, the first-order problem admits the static solution

$$\theta^{(1)} = 0.0003\,\text{rad}, \qquad \phi^{(1)} = 0, \qquad F_E^{(1)} = -0.1676\,\text{lb}$$
$$\delta_a^{(1)} = 0, \qquad \delta_e^{(1)} = -0.0021\,\text{rad}, \qquad \boldsymbol{\xi} = [4.9616 \quad 0]^T \tag{68}$$

We propose to solve the extended perturbation problem in discrete time, as described in the preceding section, under the assumption that $A(k)$, $B(k)$, and $\boldsymbol{F}_{\text{ext}}(k)$ are all constant over the sampling period between $kT$ and $(k+1)T$. When the LQR method is used, the control gain matrices $G(k)$ are determined so that the closed-loop poles are stable. The simulation of the discrete-time first-order response is carried out by means of Eqs. (55). To compute $G(k)$ by the LQR method, we use the performance measure weighting matrices

$$Q = \mathrm{diag}[1 \quad 1 \quad \cdots \quad 1 \quad 0 \quad 0 \quad \cdots \quad 0]$$
$$R = \mathrm{diag}[1 \quad 100 \quad 100 \quad 100] \tag{69}$$

To compute $A(k)$, we assume control gain coefficients $k_i = c_i = 2$ for odd $i$ and $k_i = c_i = 1$ for even $i$. The sampling period is assumed

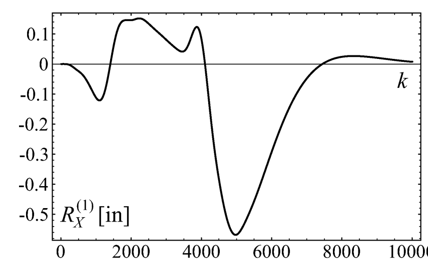


**Fig. 7 First-order longitudinal displacement.**

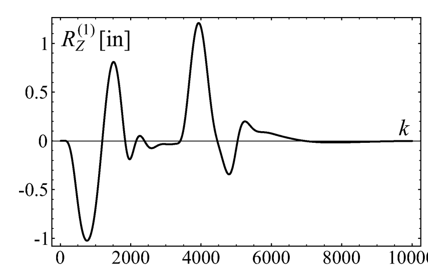


**Fig. 8 First-order vertical displacement.**

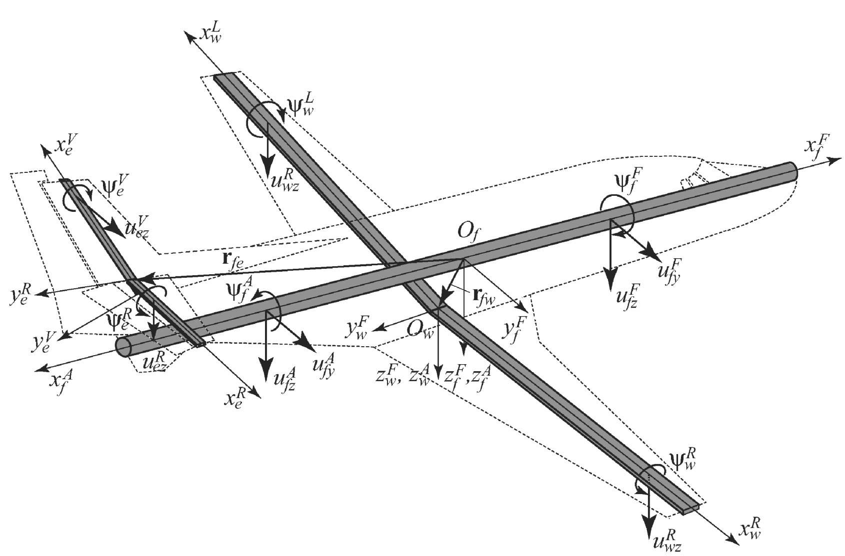


**Fig. 6 Structural model.**

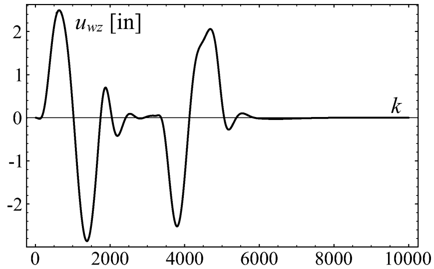


**Fig. 9 Wing-tip displacement.**

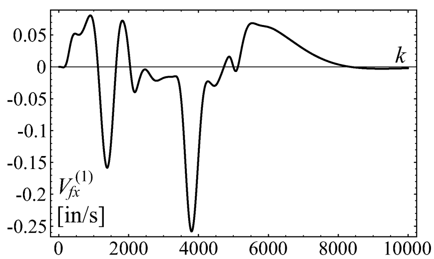


**Fig. 10 First-order forward velocity.**

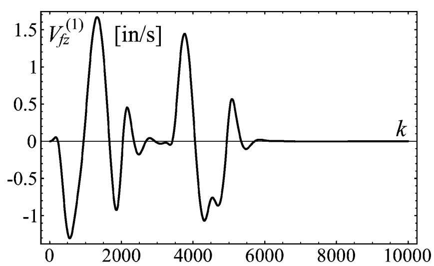


**Fig. 11 First-order plunge velocity.**

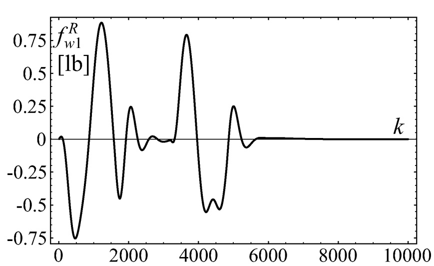


**Fig. 12 First point force on wing.**

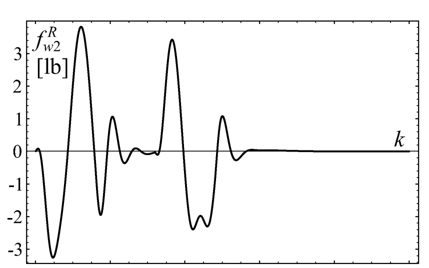


**Fig. 13 Second point force on wing.**

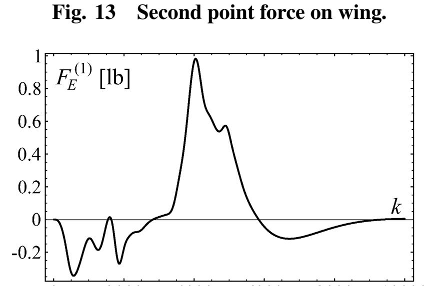


**Fig. 14 First-order engine thrust.**

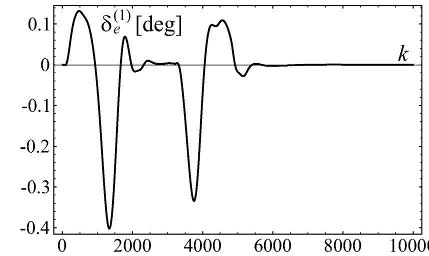


**Fig. 15 First-order elevator angle.**

to be $T = 0.0025$ s. Figures 7 and 8 show the aircraft rigid-body displacements, Fig. 9 the wing-tip elastic displacement and Figs. 10 and 11 the rigid-body velocities. Moreover, Figs. 12 and 13 show the control forces due to point actuators and Figs. 14 and 15 show the engine thrust and the elevator angle. Note that all the pertinent first-order quantities are measured from the static values given by Eqs. (68). As can be seen from the discrete-time simulations, all the solutions behave as expected.

## VIII. Conclusions

A new generation of aircraft, namely, UAVs, and, in particular, autonomous UAVs, are expected to carry out tight maneuvers. Even commercial aircraft must perform critical maneuvers occasionally, such as in collision avoidance. A recently developed theory for the dynamics and control of maneuvering flexible aircraft is ideally suited for treating such problems. The state equations describing these problems are generally nonlinear. Given the way in which aircraft are flown, a perturbation approach permits separation of the problem into one for the large rigid-body motions of the aircraft on a given flight path, referred to as the quasi-rigid flight dynamics problem, and a second one for small perturbations in the rigid-body variables and the elastic displacements, referred to as an extended perturbation problem, in which the second problem receives inputs from the solution of the first. When the aircraft executes a time-dependent maneuver, the extended perturbation equations are time varying, which tends to complicate the state equations and the corresponding control design. This paper is concerned with just such time-varying systems, and it contains several developments designed to better handle these systems. In particular, it contains an explicit derivation of the matrices defining the state equations, a new approach to the control of the perturbations from the flight path, and a formulation for computer simulations of the system response in discrete time. Computer solutions of the closed-loop state equations for the system response can be carried out conveniently by the use of MATLAB and MATHEMATICA. A numerical example illustrates the approach for a flexible aircraft carrying out a pitch maneuver. The example includes a fair number of response simulations in discrete time.

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